Utility maximization perfect complements

Problem Statement

Consider a monopolist who faces a linear market demand function given by:

$$P = a - bQ, \tag{1}$$

where P is the price, Q is the quantity produced, and a and b are positive constants. The monopolist has a linear cost function of the form:

$$C(Q) = cQ + F, \tag{2}$$

where c is the constant marginal cost and F is the fixed cost. Find the profit-maximizing quantity, price, and the maximum profit for the monopolist.

Solution

Profit function

The monopolist's profit function is given by revenue minus cost:

$$\pi(Q) = PQ - C(Q) = (a - bQ)Q - (cQ + F). \tag{3}$$

Optimal quantity

To find the profit-maximizing quantity, we differentiate the profit function with respect to Q and set the result to zero:

$$\frac{d\pi(Q)}{dQ} = a - 2bQ - c = 0. \tag{4}$$

Solving for the optimal quantity Q^*, we get:

$$Q^* = \frac{a-c}{2b}. \tag{5}$$

Optimal price

Substituting the optimal quantity Q^* back into the demand function, we can find the optimal price P^*:

$$P^* = a - bQ^* = a - b\left(\frac{a-c}{2b}\right). \tag{6}$$

Simplifying the expression, we get:

$$P^* = \frac{a+c}{2}. \tag{7}$$

Maximum profit

Finally, we substitute the optimal quantity Q^* and price P^* back into the profit function to find the maximum profit:

$$\pi^* = P^*Q^* - C(Q^*) = \left(\frac{a+c}{2}\right)\left(\frac{a-c}{2b}\right) - \left(c\left(\frac{a-c}{2b}\right) + F\right). \tag{8}$$

Simplifying the expression, we get:

$$\pi^* = \frac{(a-c)^2}{4b} - F. \tag{9}$$

Thus, the profit-maximizing quantity for the monopolist is $Q^* = \frac{a-c}{2b}$, the optimal price is $P^* = \frac{a+c}{2}$, and the maximum profit is $\pi^* = \frac{(a-c)^2}{4b} - F$.